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DYNAMIC SHAKEDOWN ANALYSIS AND BOUNDS FOR ELASTOPLASTIC STRUCTURES WITH NONASSOCIATIVE, INTERNAL VARIABLE CONSTITUTIVE LAWS†

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Abstract—Reference is made to discrete (finite element) structural models formulated in terms of generalized variables. The constitutive laws adopted are elastic–plastic, nonlinear-hardening with internal variables and generally nonassociative. The notion of *reduced domain* is employed as a generalization of a similar concept introduced in earlier developments of shakedown theory. On this basis, a unified theory is presented, which encompasses: necessary and, separately, sufficient conditions for shakedown by a static approach as a further generalization of classical Melan's theorem; bounds on various post-shakedown quantities; sufficient and necessary shakedown criteria by a kinematic approach as a further extension of Neal–Symonds–Koiter theorem. By suitable specializations and relaxations of the achievable results, the criteria and bounding inequalities established here are formulated as mathematical programming problems in view of numerical applications.

1. INTRODUCTION

Since its origin in the pioneering work of Bleich and Melan which preceded and anticipated rigid-plastic limit analysis, shakedown theory represents a prototype of *simplified* methodology: in the sense that laborious time-stepping inelastic analyses are avoided in assessing whether plastic deformations will or will not eventually cease to grow in a structure subjected to variable loads unboundedly repeated in time [e.g. see Koiter (1960) and Martin (1975)]. At an early stage the development of shakedown (SD) theory aimed at relaxing the original limitations to infinitesimal deformations, quasi-static processes and perfectly-plastic associative material models. The earliest extensions concerned dynamics (Ceradini, 1969, 1980; Corradi and Maier, 1973, 1974); nonassociative plasticity (Maier, 1969); hardening behaviour (Ponter, 1975a), also combined with geometric effects (Maier, 1973a), and were mostly based on *piece-wise-linear* constitutive models (multiple linearized yield functions) in order to exploit the potentialities of linear programming theories and methods, then fashionable topics in applied mathematics [e.g. Corradi and Zavelani (1974)].

Classical shakedown analysis and its extensions have soon been supplemented by other simplified methods in the above sense, namely by procedures intended to provide further information in terms of bounds (primarily upper bounds) on history-dependent quantities (Ponter, 1972; Maier, 1973b; Maier and Vitiello, 1974).

Meaningful later contributions are partly surveyed, e.g. in Gokhfeld and Cherniavsky (1980), König and Maier (1981) and König, (1987) and are only mentioned here through some representative references. These contributions: provided unified frameworks for both shakedown criteria and bounds (Débordes and Nayroles, 1976; Polizzotto, 1982); covered the effects of large deformations (Siemaszko and König, 1985; Weichert, 1986; Maier *et*

†Dedicated to Professor Leo Finzi on the occasion of his 70th anniversary.

al., 1993 ; Stumpf, 1993) ; proposed new bounding techniques (Capurso, 1979 ; Polizzotto, 1984a, b ; Martin, 1985 ; Polizzotto, 1986) ; developed computational approaches to shakedown analysis (König and Kleiber, 1978 ; Morelle and Nguyen, 1983 ; Kleiber and König, 1984 ; Carter and Ponter, 1986 ; Genna, 1988 ; Pycko and Mróz, 1992 ; Pycko, 1994) ; applied the theory to real-life situations and compared it to experiments (Tin-Loi, 1980 ; Alwis and Grundy, 1985 ; Lears *et al.*, 1985 ; Tin-Loi and Vimonsatit, 1993) ; finally, engineering motivations fostered generalizations to more and more versatile and realistic material models (Maier, 1987 ; Maier and Novati, 1990a, b ; Comi and Corigliano, 1991 ; Polizzotto *et al.*, 1991 ; Stein *et al.*, 1992 ; Nayroles and Weichert, 1993). The last trend has also been suggested by particular but meaningful applications of SD theory to soil-structure interaction in offshore engineering (Pande, 1982 ; Haldar *et al.*, 1990) and to steel structures subjected to thermal cycles in nuclear engineering (Morelle and Fonder, 1987 ; Save *et al.*, 1991 ; White, 1992).

The last mentioned trend is further pursued in this paper focusing on nonassociative elastoplastic models. The inelastic behaviour of technically important materials with internal friction (such as concrete and geomaterials), to many engineering purposes, admits a phenomenological description in terms of elastic–plastic constitutive laws only provided these laws are nonassociative. In fact, e.g. for concrete, the normality rule applied to Drucker–Prager's popular yield criterion would entail dilatancy grossly in excess with respect to the experimental evidence. As pointed out in a parallel paper by Pycko and Maier (in press), nonlinear hardening constitutive models now widely used for metals exhibit normality in the stress and strain spaces superposed, but not in the augmented spaces superposed (i.e. of measurable and internal static and kinematic variables) and, hence, require an *ad hoc* generalized shakedown theory.

The early generalization of the static (Melan's) theorem to nonassociative perfect plasticity (Maier, 1969), was centered on the notion of *reduced elastic domain*. A similar notion, under the more suggestive name of *elastic sanctuary* has been re-proposed very recently in a less restrictive constitutive context (Nayroles and Weichert, 1993) and, to limit analysis purposes, had been used earlier by Radenkovic (1961), De Josselin de Jong (1964), Palmer (1966) and Sacchi and Save (1968).

The contributions gathered in the present paper, though restricted to the small deformation range, are intended to provide a systematic extension to nonassociative plasticity of most of the *elastic* shakedown and bounding theory at its present (1993) stage of development.

Therefore, three preliminary choices have been made to characterize the present study : (a) dynamic context ; (b) finite element semidiscretization in generalized variables ; (c) internal variables multimode elastoplastic constitutive laws ; (d) unifying theoretical framework encompassing as special cases a number of available results. Cyclic plasticity or *plastic* shakedown is regarded here as a case of lack of SD (inadaptation) and is not dealt with *per se*. A comprehensive theory of plastic shakedown was developed recently by Polizzotto (1993).

The chosen setting is intended to help reconciling the conflicting requirements of general theoretical foundations and practical analysis methodology. In particular the present contributions aim at narrowing the persistent gap between shakedown theory and its potential applications to geodynamics and to seismic and offshore engineering, [e.g. see Pande *et al.* (1980), Aboustit and Reddy (1980) and Haldar *et al.* (1990)] ; in fact, in these areas both inertial effects and nonassociativity may play an important role.

The contents of the paper can be outlined as follows : Section 2 is devoted to a description of the two main unifying ingredients of the subsequent developments : discrete structural models centered on the notion of *generalized variables* ; material models with internal variables and with yield functions and plastic potentials, which are generally different but both subjected to the restrictions which are felt to be necessary for a comprehensive shakedown theory. The concept of generalized variables, though stemming from Prager's work, entails various advantages in multifield (mixed) finite element modeling of elastoplastic solids [e.g. see Corradi (1978), Simo *et al.* (1989), Comi *et al.* (1992) and Comi and Perego (in press)], but in addition provides here the unifying benefit of preserving essential features of the continuum tensorial description. The constitutive laws are formulated in generalized variables for materials (Gauss points), finite elements or structural components alike, and expressed in pairs of work-related internal and measurable or external variables (one *extensive* or kinematic, the other *intensive* or static), according to the general thermodynamically based pattern which is now popular in the plasticity literature (Halphen and Nguyen, 1975; Lemaitre and Chaboche, 1990).

Section 3 establishes an inequality which is called *central* or *fundamental*, inasmuch both shakedown theorems by the static approach and bounds are derived from it in the subsequent two Sections. Use is made of fictitious linear elastodynamic responses (a traditional notion in SD theory) and of the unifying concept of perturbation variables proposed by Polizzotto (1982).

Section 4 expounds results established here by the *static approach*, in the spirit of the Bleich–Melan theorem, as a further extension of Ceradini's theorem on dynamic shakedown [e.g. see Ceradini (1969), Maier (1970), Maier and Novati (1990b) and Comi and Corigliano (1991)]. A parallel generalization in the quasi-static range with reference to specific material models (Chaboche model for metals and Resende Martin's model for geomaterials) is presented in a companion paper (Pycko and Maier, in press).

Various history-dependent, post-shakedown quantities of practical interest are shown in Section 5 to be bounded from above by means of inequalities derived from the fundamental one, simply by making special selections of the perturbation variables. Bounds may turn out to be loose: some prospects of their optimization are briefly discussed.

Section 6 presents results arrived at here by a *kinematic approach*, in the spirit of Neal–Symonds' and Koiter's theorem (Symonds and Neal, 1951; Koiter, 1956), as further generalization of Corradi–Maier theorem on dynamic shakedown (Corradi and Maier, 1973, 1974; Comi and Corigliano, 1991; Polizzotto *et al.*, 1993).

In Sections 4 and 5 various contributions are presented through suitable specializations and/or relaxations of constraints, in order to formulate procedures of computational interest, mostly by (nonlinear) mathematical programming. The simplifications achievable in the practically important case of periodic excitation are pointed out, similar to those noted earlier in narrower contexts (Gavarini, 1969; Maier and Novati, 1990b).

Finally in Section 7 the main findings are synthesized with supplementary comments.

2. FORMULATION OF PROBLEM

2.1. *Governing relations in generalized variables*

The inelastic, small deformation dynamic response of a solid or structure occupying volume Ω and modeled in space as an aggregate of finite elements or structural components can be governed by (ordinary, nonlinear) differential equations of the following kind:

$$\boldsymbol{\varepsilon}(t) = \mathbf{C}\mathbf{u}(t) \tag{1}$$

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{V}\dot{\mathbf{u}}(t)+\mathbf{C}^{\mathrm{t}}\boldsymbol{\sigma}(t) = \mathbf{P}(t) \tag{2}$$

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}(\dot{\boldsymbol{\varepsilon}};\boldsymbol{\varepsilon}(\tau),\tau \leqslant t)) \tag{3}$$

associated to the initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0. \tag{4}$$

The symbology used above is specified as follows: $\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are vectors of the *generalized* variables which govern the (unknown) displacements, strains and stresses, respectively, through the relevant, element-wise chosen interpolations; $\mathbf{P}$ denotes the load vector equivalent to the given external forces; $\mathbf{C}$ is the geometric compatibility matrix; $\mathbf{M}$ and $\mathbf{V}$ represent the (symmetric, consistent) matrices of inertia and viscous damping, the former being assumed positive-definite, the latter positive-semidefinite; finally, dots mark derivatives with respect to time t (or τ). Note that the linear damping defined by matrix **V** expresses structural (e.g. frictional) effects and not material viscosity. **V** can be assumed as linear combination of the mass and the elastic stiffness matrices.

Nonlinearity is implied only by the elastic–plastic (inviscid) constitutive law, which will be explicitly formulated later in an internal variable format.

The variables used in the semidiscretization leading to eqns (1)–(4) are called *generalized* in Prager's sense when they are endowed of the following features: (i) variables are introduced in pairs of conjugate quantities, each pair containing a *static* field, say $\hat{\mathbf{s}}$ ($\mathbf{x}$) (or vector $\mathbf{s}$) and a *kinematic* field, say $\hat{\mathbf{k}}$ ($\mathbf{x}$) (or vector $\mathbf{k}$), $\mathbf{x}$ being space coordinates; (ii) denoting by $\mathbf{N}_s$ and $\mathbf{N}_k$ *shape matrices* of interpolation functions, such that:

$$\hat{\mathbf{s}}(\mathbf{x}) = \mathbf{N}_s(\mathbf{x})\mathbf{s} \quad \hat{\mathbf{k}}(\mathbf{x}) = \mathbf{N}_k(\mathbf{x})\mathbf{k} \tag{5}$$

the (*inverse*) relations hold true:

$$\mathbf{s} = \int_\Omega \mathbf{N}_k^T(\mathbf{x})\hat{\mathbf{s}}(\mathbf{x})\,d\Omega \quad \mathbf{k} = \int_\Omega \mathbf{N}_s^T(\mathbf{x})\hat{\mathbf{k}}(\mathbf{x})\,d\Omega. \tag{6}$$

The meaning and implications of the recourse to *generalized variables* in the construction of a semidiscrete structural model can be elucidated through the following remarks.

(a) A necessary and sufficient condition for eqns (6) to hold as a consequence of interpolations (5) is provided by the orthogonality condition of the conjugate interpolation functions:

$$\int_\Omega \mathbf{N}_s^T(\mathbf{x})\mathbf{N}_k(\mathbf{x})\,d\Omega = \mathbf{I}. \tag{7}$$

(b) A crucial implication of eqs. (6) is the conservation of the scalar product, namely, the relation:

$$\int_\Omega \hat{\mathbf{s}}^T(\mathbf{x})\hat{\mathbf{k}}(\mathbf{x})\,d\mathbf{\Omega} = \mathbf{s}^T\mathbf{k} \quad \text{for any} \quad \mathbf{s}, \mathbf{k}. \tag{8}$$

(c) In order to satisfy relation (7), a simple way is to choose first interpolation functions of one field, e.g. $\mathbf{N}_k(\mathbf{x})$, and then to determine the interpolation functions of the conjugate one, $\mathbf{N}_s(\mathbf{x})$ through the relationship:

$$\mathbf{N}_s(\mathbf{x}) = \left(\int_\Omega \mathbf{N}_k^T(\mathbf{x})\mathbf{N}_k(\mathbf{x})\,d\Omega\right)^{-1} \mathbf{N}_k(\mathbf{x}). \tag{9}$$

(d) Eqns (5)–(9) can be written for each element Ω^e (using an element index e on all symbols). As an alternative, here preferred, each interpolation can be defined over the whole solid or structure (i.e. with $\mathbf{x} \in \Omega$), being understood that it vanishes outside its *support*, so that vectors **s** and **k** concern the assemblage of elements.

(e) The above definitions and remarks apply to the following kinematic (or *extensive*) variables: strains (total $\boldsymbol{\varepsilon}$, elastic $\mathbf{e}$ and plastic $\mathbf{p}$), kinematic internal variables $\boldsymbol{\eta}$, plastic multipliers $\boldsymbol{\lambda}$. The conjugate static (or *intensive*) variables are, respectively: stresses $\boldsymbol{\sigma}$, static internal variables $\boldsymbol{\chi}$, yield functions $\boldsymbol{\varphi}$ (and plastic potentials $\boldsymbol{\Phi}$). The pairs $(\boldsymbol{\eta}, \boldsymbol{\chi})$ and $(\dot{\boldsymbol{\lambda}}, \boldsymbol{\varphi})$ will be defined below (Section 2.2) by making the constitutive law (3) explicit. Another pair is formed by displacements **u** and loads **P**.

(f) Special cases of *consistent* modeling by generalized variables are discussed elsewhere (Comi and Perego, in press). In order to fix ideas, one can refer here, e.g. to a 3D solid discretized into constant-strain, four node, tetrahedral described in terms of *natural* element variables ($\varepsilon \equiv$ edge elongations; $\boldsymbol{\sigma} \equiv$ equivalent self equilibrated edge forces).

Imposed (e.g. thermal) strains might be introduced in the constitutive law (3) but will not be considered here for brevity. Similarly, possible imposed displacements will be assumed zero (they may always be simulated by suitable strains imposed on fictitious elements added along on the kinematically constrained variables).

2.2. *Elastic–plastic nonassociative constitutive laws with internal variables*

The above adopted semidiscretization based on the notion of generalized variables implies, as a first formal advantage, that constitutive laws preserve a meaning and all essential features in passing from material (i.e. homogeneous specimen; or Gauss point) to individual finite elements and to assembled aggregates of finite elements. We will profit from this circumstance by formulating the envisaged class of material models directly in terms of the generalized strain and stress vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ which already intervened in eqns (1)–(4).

The Helmholtz's *free energy* is assumed to be the sum of two addends: the recoverable elastic strain energy Ψ_e expressed by a positive definite quadratic form of the elastic strains $\mathbf{e}$; the *stored* elastic energy Ψ_s, locked in the material by the microscale rearrangements globally described by the kinematic internal variables $\boldsymbol{\eta}$ and expressed as a convex, differentiable function of them:

$$\Psi(\mathbf{e}, \boldsymbol{\eta}) = \tfrac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{E}\mathbf{e} + \Psi_s(\boldsymbol{\eta}), \tag{10}$$

where $\mathbf{E}$ represents the (block-diagonal, symmetric, positive definite) matrix of the elastic stiffnesses.

The static counterparts $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ are related to the kinematic variables $\mathbf{e}$ and $\boldsymbol{\eta}$, respectively, as gradients of the potential Ψ, i.e. through the state equations:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \mathbf{e}} = \mathbf{E}\mathbf{e}; \quad \boldsymbol{\chi} = \frac{\partial \Psi}{\partial \boldsymbol{\eta}} = \frac{\partial \Psi_s}{\partial \boldsymbol{\eta}}. \tag{11}$$

The former equation is Hooke's law, the latter defines the (generally nonlinear) hardening rule. The plastic potentials, collected in vector $\boldsymbol{\Phi}(\boldsymbol{\sigma}, \boldsymbol{\chi})$, are assumed to be convex and differentiable functions of the static variables $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$. If vector $\dot{\boldsymbol{\lambda}}$ gathers the plastic multiplier rates measuring the yielding processes in all modes of all elements, the flow rules which govern the consequent evolution of plastic strains $\mathbf{p}$ and kinematic internal generalized variables $\boldsymbol{\eta}$, are defined by:

$$\dot{\mathbf{p}} = \frac{\partial \boldsymbol{\Phi}^{\mathrm{T}}}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\eta}} = -\frac{\partial \boldsymbol{\Phi}^{\mathrm{T}}}{\partial \boldsymbol{\chi}}\dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}. \tag{12}$$

The mechanical dissipation has to comply with the thermodynamical sign-constraint:

$$\dot{D} \equiv \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\Psi} = \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - (\boldsymbol{\sigma}^{\mathrm{T}}\dot{\mathbf{e}} + \boldsymbol{\chi}^{\mathrm{T}}\dot{\boldsymbol{\eta}}) = \boldsymbol{\sigma}^{\mathrm{T}}\dot{\mathbf{p}} - \boldsymbol{\chi}^{\mathrm{T}}\dot{\boldsymbol{\eta}} \geqslant 0, \tag{13}$$

where the assumed additivity of elastic $\mathbf{e}$ and plastic $\mathbf{p}$ strains has been used.

The yield functions gathered in vector $\boldsymbol{\varphi}$ $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ are for all modes assumed convex differentiable functions of the static variables, so that they define, locally for all elements separately, convex *fixed* yield domains in the spaces of element generalized stresses and static internal variables (called henceforth *augmented* stress spaces):

$$\varphi(\sigma, \chi) \leqslant \mathbf{0}. \tag{14}$$

The loading–unloading conditions (typical of plastic models centered on the concept of the yield surface) are expressed by the usual complementarity relationship, which holds also component wise due to the sign-constraints on the two vectors involved:

$$\varphi^{T}\dot{\lambda} = 0. \tag{15}$$

Since in the above constitutive law the plastic potentials play a role only through their gradients, the indeterminate additive constants will henceforth be removed by setting, as a *normalization*, that they are equal in the origin to the yield functions.

It is important to notice that the theoretical results presented in the subsequent sections are derived under the hypothesis that neither saturation hardening nor bounding surfaces are considered in the class of constitutive models here discussed. Generalization of shakedown conditions, in particular for the kinematic approach (see Section 6) for constitutive laws in which saturation hardening and bounding surfaces are taken into account are presented in a parallel paper (Corigliano *et al.*, in press).

2.3. *Restrictions on the nonassociativity.*

Focus is herein on constitutive models which entail plastic potentials $\mathbf{\Phi}$ different from the yield functions φ pertaining to the same yield modes.

This difference (i.e. *nonassociativity*, or lack of normality) is strongly suggested by the experimentally observed behaviour of many engineering materials as the main manifestation of their internal friction, as noted in Section 1. It is well known that nonassociativity has far-reaching consequences in plasticity theory. First of all it generally invalidates Drucker's classical postulate (Drucker, 1964). This is often given the formulation:

$$\int_{\sigma}^{\sigma'} [\sigma(t)-\sigma']^{T}\,\mathrm{d}\mathbf{p}(t) \geqslant 0, \quad \forall\sigma' \text{ such that } \varphi(\sigma') \leqslant \mathbf{0}. \tag{16}$$

Its main particular implications read:

$$[\sigma(t)-\sigma']^{T}\dot{\mathbf{p}}(t) \geqslant 0 \quad \forall\sigma' \text{ such that } \varphi(\sigma') \leqslant 0; \quad \dot{\sigma}^{T}\dot{\varepsilon} \geqslant 0, \forall\dot{\varepsilon}. \tag{17}$$

Of eqns (17), the former represents Hill's maximum work principle. The latter means stability according to the second-order work statical criterion, and was shown (Maier and Hueckel, 1979), to be violated for all paths $\dot{\varepsilon}$ belonging to a cone whenever the hardening modulus h is below a non-negative threshold h_c. In perfect plasticity, i.e. for $h = 0$, nonassociativity implies $h_c > 0$ and, hence, lack of stability according to the criterion (17).

In order to later establish shakedown criteria, the following hypothesis will be used in combination with the constitutive relations (10)–(15).

$$(\mathbf{\Phi}-\mathbf{B})^{T}\dot{\lambda} \geqslant 0. \tag{18}$$

Here $\mathbf{\Phi}$ is the vector of plastic potentials (see eqn (12); $\dot{\lambda}$ is the vector of plastic multipliers. The vector of constants $\mathbf{B}$ is defined (and numerically evaluated) as follows:

$$B_{\alpha} = \min_{\sigma,\chi} \Phi_{\alpha}(\sigma, \chi) \tag{19a}$$

subject to:

$$\varphi_{\alpha}(\sigma, \chi) = 0, \quad \varphi_{\beta}(\sigma, \chi) \leqslant 0, \quad \forall\beta \neq \alpha. \tag{19b}$$

Problem (19) amounts to minimizing the α-th plastic potential Φ_{α} in the augmented space (σ, χ) over the corresponding α-th yield mode, namely over the portion of the yield surface

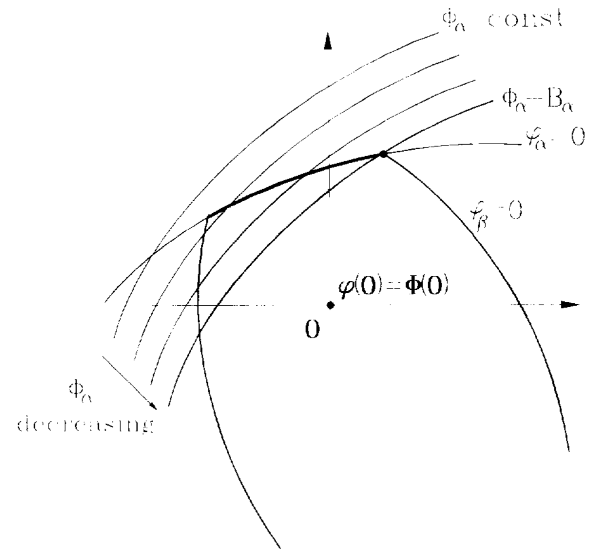

Fig. 1. Generation of the α-th mode of the reduced domain schematically illustrated in the augmented space $\boldsymbol{\sigma}, \boldsymbol{\chi}$: shaded region out of reduced domain.

defined by the equation $\varphi_\alpha(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ (see Fig. 1). The constants B_α generated by (locally) solving problems (19) for all α, are collected in vector $\mathbf{B}$ and give rise to the inequality:

$$\boldsymbol{\Phi}(\boldsymbol{\sigma}, \boldsymbol{\chi}) - \mathbf{B} \leqslant \mathbf{0}. \tag{20}$$

The convex domain defined by (20) will be referred to as *reduced domain*. If $\mathbf{B}$ is defined as above, relation (18) is certainly satisfied because, whenever $\dot{\lambda} > 0$, the corresponding point $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ in the space $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ belongs to the yield surface and therefore is outside, or at most on, the boundary of the reduced domain [i.e. $\boldsymbol{\Phi}(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) - \mathbf{B} \geqslant 0$]. The definition of the vector of constants $\mathbf{B}$ in the form of a constrained optimization problem (19), of the reduced domain and related notions are discussed and illustrated by special cases in (Pycko and Maier, in press).

Inequality (18) is seen to be complied with if the difference between the plastic potentials and the yield functions is bounded from below over the yield domain, i.e.

$$\boldsymbol{\Phi} - \boldsymbol{\varphi} \geqslant \mathbf{B}. \tag{21}$$

Inequality (18) can thus easily be proved from (21) through equation (15). As a special case associative behaviour, i.e. $\boldsymbol{\Phi} = \boldsymbol{\varphi}$, implies $\mathbf{B} = \mathbf{0}$ in (18).

The existence of (finite) constants $\mathbf{B}$ and, hence, the validity of inequality (18) can be regarded as a (weak) constitutive hypothesis additional to (10)–(15).

2.4. *Shakedown and related concepts*

Like in classical plasticity theory, shakedown (or *elastic* shakedown or *adaptation* or *stabilization*) will be said to occur in a dynamical system modeled as described in what precedes, if a suitable overall cumulative (non decreasing) measure of the yielding process is bounded above in time. Such measure is identified here in the energy dissipated throughout the structural model. In other terms in view of eqn (13), the shakedown criterion adopted reads (τ being integration variable):

$$\lim_{t \to \infty} \left\{ D(t) = \int_0^t (\boldsymbol{\sigma}^{\mathrm{T}} \dot{\mathbf{p}} - \boldsymbol{\chi}^{\mathrm{T}} \dot{\boldsymbol{\eta}})\, \mathrm{d}\tau \right\} < \infty. \tag{22}$$

Inadaptation, i.e. the event contrary to shakedown, ($D(t) \to \infty$) occurs either with unbounded displacements ($\|\mathbf{u}\| \to \infty$: *incremental collapse* or *ratchetting*) or with bounded displacements and, hence, bounded plastic strains (*alternating plasticity*).

The next two Sections are intended to establish *a priori* criteria for SD or lack thereof. *A priori* means here susceptible to be used on the basis of purely linear-elastic analysis of a suitably defined *fictitious process*, thus avoiding laborious inelastic time-stepping solutions. In the same sense *a priori* upper bounds will also be proven on post-shakedown, history-dependent quantities.

3. FICTITIOUS ELASTIC PROCESSES AND CENTRAL INEQUALITY

3.1. *Linear-elastic auxiliary analyses*

In view of subsequent developments, let us consider the following problems concerning the structural model of Section 2 now supposed to be uncapable of any plastic yielding.

(A) Elasto-dynamic response (superscript E) to the given external actions $\mathbf{P}(t)$ with homogeneous initial conditions. The governing equations are:

$$\mathbf{M}\ddot{\mathbf{u}}^{E}(t)+\mathbf{V}\dot{\mathbf{u}}^{E}(t)+\mathbf{C}^{T}\boldsymbol{\sigma}^{E}(t) = \mathbf{P}(t) \tag{23a}$$

$$\boldsymbol{\varepsilon}^{E} = \mathbf{C}\mathbf{u}^{E}, \quad \boldsymbol{\sigma}^{E} = \mathbf{E}\boldsymbol{\varepsilon}^{E}, \quad \mathbf{u}^{E}(0) = \mathbf{0}, \quad \dot{\mathbf{u}}^{E}(0) = \mathbf{0}. \tag{23b}$$

(B) Free vibration (superscript F) owing to suitably chosen, generally fictitious (capped symbols) initial conditions in the absence of external loads:

$$\mathbf{M}\ddot{\hat{\mathbf{u}}}^{F}(t)+\mathbf{V}\dot{\hat{\mathbf{u}}}^{F}(t)+\mathbf{C}^{T}\hat{\sigma}^{F}(t) = \mathbf{0} \tag{24a}$$

$$\hat{\boldsymbol{\varepsilon}}^{F} = \mathbf{C}\hat{\mathbf{u}}^{F}, \quad \hat{\boldsymbol{\sigma}}^{F} = \mathbf{E}\hat{\boldsymbol{\varepsilon}}^{F}; \quad \hat{\mathbf{u}}^{F}(0) = \hat{\mathbf{u}}_{0}, \quad \dot{\hat{\mathbf{u}}}^{F}(0) = \dot{\hat{\mathbf{u}}}_{0}. \tag{24b}$$

(C) Elastostatic selfstress response $\hat{\rho}$ to a time-independent plastic strain distribution $\hat{\mathbf{p}}^{s}$. The relevant governing equations

$$\mathbf{C}^{T}\hat{\rho} = \mathbf{0}, \quad \boldsymbol{\varepsilon}^{s} = \mathbf{C}\hat{\mathbf{u}}^{s}, \quad \hat{\rho} = \mathbf{E}(\hat{\boldsymbol{\varepsilon}}^{s}-\hat{\mathbf{p}}^{s}) \tag{25}$$

can be solved explicitly, whenever convenient, to give:

$$\hat{\rho} = \mathbf{Z}\hat{\mathbf{p}}^{s}, \quad \text{where } \mathbf{Z} = \mathbf{EC}(\mathbf{C}^{T}\mathbf{EC})^{-1}\mathbf{C}^{T}\mathbf{E}-\mathbf{E}. \tag{26}$$

Clearly, the solution to the elastodynamic problem (A) captures the loading history data, and can be regarded henceforth as an input for subsequent inelastic analyses.

3.2. *A fundamental inequality*

The subsequent derivation of shakedown criteria and bounds can be carried out in a concise way, if it is based on the following statement (Prop. 1), which *per se* does not exhibit an explicit mechanical meaning (Polizzotto, 1982).

Consider a *fictitious process* (or *comparison elastic response*) consisting of the superposition of the solutions to the linear problems A–C, of Section 3.1, eqns (23–25) and denoted henceforth by a cap (without superscript) on the symbols of the relevant quantity, e.g.

$$\hat{\boldsymbol{\sigma}}(t) \equiv \boldsymbol{\sigma}^{E}(t)+\hat{\boldsymbol{\sigma}}^{F}(t)+\hat{\rho}. \tag{27}$$

Proposition 1. *Fundamental inequality*. Suppose by hypothesis that:

$$\mathbf{\Phi}((\hat{\boldsymbol{\sigma}}(t)+\mathbf{r})q,(\hat{\boldsymbol{\chi}}+\mathbf{v})s)+\mathbf{w}\leqslant\mathbf{B},\quad \text{at any } t\geqslant\hat{t} \tag{28}$$

for a certain finite time instant $\hat{t}$, for some parameters q, s, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{w}$, internal variables $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\chi}}$, self stresses $\hat{\boldsymbol{\rho}}$, (all time-independent) and, finally, for some fictitious initial conditions giving rise to the free vibration stresses $\hat{\boldsymbol{\sigma}}^{\mathrm{F}}(t)$.

Then (as a thesis proven below) the following inequality holds true:

$$\int_{\hat{t}}^{t}[(\boldsymbol{\sigma}+\mathbf{r})^{T}\dot{\mathbf{p}}q-(\boldsymbol{\chi}+\mathbf{v})^{T}\dot{\boldsymbol{\eta}}s]\,\mathrm{d}\tau-\int_{\hat{t}}^{t}\dot{D}(\tau)\,\mathrm{d}\tau+\mathbf{w}^{\mathrm{T}}\int_{\hat{t}}^{t}\dot{\boldsymbol{\lambda}}\,\mathrm{d}\tau\leqslant qL_{\mathrm{e}}(\hat{t})+sL_{\mathrm{s}}(\hat{t}), \tag{29a}$$

where

$$L_{\mathrm{e}}(t)\equiv\tfrac{1}{2}\Delta\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{M}\Delta\dot{\mathbf{u}}+\tfrac{1}{2}\Delta\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{E}^{-1}\Delta\boldsymbol{\sigma} \tag{29b}$$

$$L_{\mathrm{s}}(t)\equiv\Psi_{\mathrm{s}}(\boldsymbol{\eta})-\Psi_{\mathrm{s}}(\hat{\boldsymbol{\eta}})-(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})^{\mathrm{T}}\hat{\boldsymbol{\chi}}. \tag{29c}$$

In eqns (29): τ is the dummy variable of time integration; Δ denotes the difference between the actual structural response and the above defined fictitious process, cf. eqn (27); $\hat{\boldsymbol{\chi}}$ is conjugated with $\hat{\boldsymbol{\eta}}$ through eqn (11b).

The time independent parameters q, s, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{w}$ in eqns (28) and (29) have no physical meaning but provide a convenient unified tool for the later generation of bounds. They can be called *gap* or *perturbation* parameters (without any link with conventional perturbation methods). The first and the second integral in eqn (29a) can be interpreted as a multi-perturbed and the actual cumulative dissipated energy (cp. eqn 13), respectively.

Proof. Since in the actual process $\dot{\boldsymbol{\lambda}}\geqslant\mathbf{0}$, inequality (28) implies:

$$\mathbf{w}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}\leqslant-(\tilde{\mathbf{\Phi}}-\mathbf{B})^{T}\dot{\boldsymbol{\lambda}}. \tag{30}$$

In the hypothesis inequality (28) the argument $(\tilde{\boldsymbol{\sigma}},\tilde{\boldsymbol{\chi}})$ of the plastic potential $\mathbf{\Phi}$ can be interpreted as *perturbed* fictitious stresses and static internal variables, namely:

$$\tilde{\boldsymbol{\sigma}}(t)\equiv(\hat{\boldsymbol{\sigma}}(t)+\mathbf{r})q,\quad \tilde{\boldsymbol{\chi}}\equiv(\hat{\boldsymbol{\chi}}+\mathbf{v})s,\quad \tilde{\mathbf{\Phi}}\equiv\mathbf{\Phi}(\tilde{\boldsymbol{\sigma}}(t),\tilde{\boldsymbol{\chi}}). \tag{31}$$

In view of the constitutive assumption (18), we can write:

$$-(\tilde{\mathbf{\Phi}}-\mathbf{B})^{\mathrm{T}}\dot{\boldsymbol{\lambda}}\leqslant(\mathbf{\Phi}-\mathbf{B})^{\mathrm{T}}\dot{\boldsymbol{\lambda}}-(\tilde{\mathbf{\Phi}}-\mathbf{B})^{\mathrm{T}}\dot{\boldsymbol{\lambda}}=(\mathbf{\Phi}-\tilde{\mathbf{\Phi}})^{\mathrm{T}}\dot{\boldsymbol{\lambda}}. \tag{32}$$

Because of the assumed convexity of the plastic potentials $\mathbf{\Phi}$ and making use of the flow rule (12) we can write:

$$(\tilde{\mathbf{\Phi}}-\mathbf{\Phi})^{\mathrm{T}}\dot{\boldsymbol{\lambda}}\geqslant(\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})^{\mathrm{T}}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\boldsymbol{\sigma}}\dot{\boldsymbol{\lambda}}+(\tilde{\boldsymbol{\chi}}-\boldsymbol{\chi})^{\mathrm{T}}\frac{\partial\mathbf{\Phi}^{\mathrm{T}}}{\partial\boldsymbol{\chi}}\dot{\boldsymbol{\lambda}}=(\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})^{\mathrm{T}}\dot{\mathbf{p}}-(\tilde{\boldsymbol{\chi}}-\boldsymbol{\chi})^{\mathrm{T}}\dot{\boldsymbol{\eta}}. \tag{33}$$

From (30), (32) and (33) it follows that:

$$\mathbf{w}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}\leqslant(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\mathbf{p}}-(\boldsymbol{\chi}-\tilde{\boldsymbol{\chi}})^{\mathrm{T}}\dot{\boldsymbol{\eta}}. \tag{34}$$

By substituting into it eqns (31a,b) and integrating over the time interval $[\hat{t},t]$, inequality (34) yields:

$$\int_{\hat{t}}^{t} [(\boldsymbol{\sigma}+\mathbf{r})^{\mathrm{T}}\dot{\mathbf{p}}q-(\boldsymbol{\chi}+\mathbf{v})^{T}\dot{\boldsymbol{\eta}}s]\,\mathrm{d}\tau-\int_{\hat{t}}^{t}(\boldsymbol{\sigma}^{\mathrm{T}}\dot{\mathbf{p}}-\boldsymbol{\chi}^{\mathrm{T}}\dot{\boldsymbol{\eta}})\,\mathrm{d}\tau$$

$$+\mathbf{w}^{\mathrm{T}}\int_{\hat{t}}^{t}\dot{\boldsymbol{\lambda}}\,\mathrm{d}\tau \leqslant q\int_{\hat{t}}^{t}(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\mathbf{p}}\,\mathrm{d}\tau-s\int_{\hat{t}}^{t}(\boldsymbol{\chi}-\hat{\boldsymbol{\chi}})^{\mathrm{T}}\dot{\boldsymbol{\eta}}\,\mathrm{d}\tau. \quad (35)$$

Focusing now on the first term on the r.h.s. of eqn (35), we write the virtual work equation:

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^{\mathrm{T}}\cdot[\dot{\mathbf{p}}+\mathbf{E}^{-1}(\dot{\boldsymbol{\sigma}}-\dot{\hat{\boldsymbol{\sigma}}})] = -(\ddot{\mathbf{u}}-\ddot{\hat{\mathbf{u}}})^{\mathrm{T}}\mathbf{M}\cdot[\dot{\mathbf{u}}-\dot{\hat{\mathbf{u}}}]-(\dot{\mathbf{u}}-\dot{\hat{\mathbf{u}}})^{T}\mathbf{V}\cdot[\dot{\mathbf{u}}-\dot{\hat{\mathbf{u}}}]. \quad (36)$$

In fact, the factors in square brackets are easily recognized to form a set of compatible kinematic quantities, while the factors which pre-multiply them turn out to satisfy the dynamic equilibrium equations. Since the viscous damping matrix **V** is positive semidefinite, without the relevant term eqn (36) becomes an inequality; which after rearrangements can be written in the form:

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\mathbf{p}} \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\Delta\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{E}^{-1}\Delta\boldsymbol{\sigma}+\tfrac{1}{2}\Delta\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{M}\Delta\dot{\mathbf{u}}]. \quad (37)$$

As for the second term on the r.h.s. of eqn (35), recalling the definition of stored-energy potential Ψ_s, eqn (10), and the state equation (11b), account taken of the time-independence of $\hat{\boldsymbol{\eta}}$ (and $\hat{\boldsymbol{\chi}}$), it leads to the equality:

$$(\boldsymbol{\chi}-\hat{\boldsymbol{\chi}})^{\mathrm{T}}\dot{\boldsymbol{\eta}} = \frac{\mathrm{d}}{\mathrm{d}t}[\Psi_s(\boldsymbol{\eta})-\Psi_s(\hat{\boldsymbol{\eta}})-(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})^{\mathrm{T}}\hat{\boldsymbol{\chi}}]. \quad (38)$$

Let us now make use of the inequality (37) and of eqn (38) in the integrands on the r.h.s. of inequality (35). Integrating in time and keeping in mind the definitions of L_e (29b) and L_s (29c), this r.h.s. can be given the expression:

$$q(L_e(\hat{t})-L_e(t))+s(L_s(\hat{t})-L_s(t)) \quad (39)$$

and can be deprived of the addends $L_e(t)$ and $L_s(t)$ without jeopardizing inequality (35), since the former addend is non-negative owing to the nature of matrices **M** and **E** and the latter is so owing to the assumed convexity of the stored energy potential Ψ_s and to the fact that $\hat{\boldsymbol{\chi}} \equiv \partial\Psi_s/\partial\boldsymbol{\eta}|_{\hat{\eta}}$. Thus, the inequality (35) is seen to reduce to the inequality (29a), which embodies the thesis to prove. (q.e.d.)

4. SHAKEDOWN ANALYSIS BY A STATIC APPROACH

4.1. *Shakedown theorems*

Static approach means here that constant static variables, namely selfstresses $\hat{\boldsymbol{\rho}}$ (like in Melan's theorem) and internal variables $\hat{\boldsymbol{\chi}}$ (which do not exist in perfect plasticity) play the role of trial parameters, besides fictitious initial condition $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$. Antecedents can be found in Melan (1938), and, as for dynamics, in Ceradini (1969) for perfect plasticity and Maier (1970) for piece-wise-linear hardening plasticity.

Let us specialize the choice of the perturbation parameters which intervene in the hypothesis (28), by setting the vectors to zero and taking scalars q and s equal:

$$\mathbf{r} = \mathbf{0};\mathbf{v} = \mathbf{0};\mathbf{w} = \mathbf{0} \quad q = s = \omega > 1. \quad (40)$$

Thus the fundamental inequality (29a), becomes:

$$\int_{\hat{t}}^{t} \dot{D}(\tau)\,\mathrm{d}\tau \leqslant \frac{\omega}{\omega-1}[L_e(\hat{t})+L_s(\hat{t})]. \tag{41}$$

Since the r.h.s. is a nonnegative finite quantity independent of the current time t, in view of the definition (22) of shakedown, inequality (41) means that the quantity $(D(t)-D(\hat{t}))$ and, hence, since $\hat{t}$ is a fixed finite time, the overall cumulative dissipated energy $D(t)$ is bounded above in time.

By recovering now the hypothesis (28) specialized in accordance with assumption (40), the conclusion attained can be stated as follows.

Proposition 2. *Sufficient condition for shakedown.* The semidiscretized (space-modeled) structure will shakedown under the given loading history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, if there exist a time $\hat{t}$, time-independent static internal variables $\hat{\boldsymbol{\chi}}$, a scalar $\omega > 1$ and a fictitious process $\hat{\boldsymbol{\sigma}}(t)$, see eqn (27), (i.e. selfstresses $\hat{\boldsymbol{\rho}}$ and fictitious initial conditions $\hat{\mathbf{u}}_0$, $\hat{\dot{\mathbf{u}}}_0$), such that:

$$\boldsymbol{\Phi}(\omega\hat{\boldsymbol{\sigma}}, \omega\hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \forall t \geqslant \hat{t}, \tag{42}$$

where $\mathbf{B}$ is the vector of constants which appears in the constitutive relationship (18).

Like in the classical theory, a supplementary conclusion flows from the notion that shakedown means that plastic yielding does not occur after a finite time $\hat{t}$. Through a direct customary path of reasoning, not duplicated here for brevity [e.g. see Maier and Novati (1990a)] the following statement is easily established.

Proposition 3. *Necessary condition for shakedown.* If the structure shakes down under the given external actions, then there exist a time $\hat{t}$, time-independent static internal variables $\hat{\boldsymbol{\chi}}$, and a fictitious process $\hat{\boldsymbol{\sigma}}(t)$, such that:

$$\boldsymbol{\varphi}(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}}) \leqslant \mathbf{0}, \quad \forall t \geqslant \hat{t}. \tag{43}$$

Adopt again the special selection (40) of the perturbation variables but with $q = s = \omega = 1$. The same path of reasoning followed in the proof of Prop. 1, Section 3.2, (except the time integration) leads first to the specialization of eqn (34), namely to:

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\mathbf{p}}-(\boldsymbol{\chi}-\hat{\boldsymbol{\chi}})^{\mathrm{T}}\dot{\boldsymbol{\eta}} \geqslant 0 \tag{44}$$

whence, through eqns (37), (38) and (29c), one arrives at the inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\Delta\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{E}^{-1}\Delta\boldsymbol{\sigma}+\tfrac{1}{2}\Delta\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{M}\Delta\dot{\mathbf{u}}+L_s(t)] \leqslant 0. \tag{45}$$

Note that, by means of a Taylor series expansion around $\hat{\boldsymbol{\eta}}$, the expression (29c) of the stored energy function L_s, becomes:

$$L_s(t) = (\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})^{\mathrm{T}}\frac{\partial^2\Psi_s}{\partial\boldsymbol{\eta}\,\partial\boldsymbol{\eta}^{\mathrm{T}}}(\hat{\boldsymbol{\eta}})(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}}) + \text{higher order terms}. \tag{46}$$

The Hessian matrix in eqn (46) is positive semidefinite because of the assumed convexity of the stored energy Ψ_s. It becomes constant for linear hardening (hardening matrix $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$), when the higher order terms in eqn (46) identically vanish. In this case if $\mathbf{A}$ is nonsingular, $L_e + L_s$ can be assumed as an energy norm of the difference between the actual and the fictitious process. As a conclusion, inequality (45) can be interpreted by the following statement.

Proposition 4. *Non-expansivity property*. If the same hypothesis of Prop. 2, inequality (42), holds with $\omega = 1$, then the distance between the actual and the fictitious process measured by the energy norm $L_e + L_s$, does not increase in time.

4.2. *Bases of analysis procedures and remarks*

The above statements, especially 2 and 3, play a crucial role as a basis for analysis procedures and, hence, are worth being commented on in some detail.

Let $\alpha \geqslant 0$ be the *load factor*, i.e. a common multiplier of all external actions. Shakedown analysis primarily seeks the *safety factor* with respect to inadaptation, or *shakedown limit* α_s, namely the critical threshold α_s below which the system still shakes down and above which it does not. The following bounding statements emanate from Props 2 and 3, respectively.

Proposition 5. *Lower bound on* α_s. The shakedown limit α_s is bounded from below by the number α_s^- such that:

$$\alpha_s \geqslant \alpha_s^- \equiv \max_{\substack{\alpha,\hat{\rho},\hat{\chi},\\ \hat{t},\hat{\mathbf{u}}_0,\dot{\hat{\mathbf{u}}}_0,}} \{\alpha\}, \quad \text{subject to:} \quad \boldsymbol{\Phi}(\alpha\boldsymbol{\sigma}^E(t)+\hat{\boldsymbol{\sigma}}^F(t)+\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \forall t \geqslant \hat{t}. \tag{47}$$

Proposition 6. *Upper bound on* α_s. The shakedown limit α_s is bounded from above by the number α_s^+ such that:

$$\alpha_s \leqslant \alpha_s^+ \equiv \max_{\substack{\alpha,\hat{\rho},\hat{\chi},\\ \hat{t},\hat{\mathbf{u}}_0,\dot{\hat{\mathbf{u}}}_0,}} \{\alpha\}, \quad \text{subject to:} \quad \boldsymbol{\varphi}(\alpha\boldsymbol{\sigma}^E(t)+\hat{\boldsymbol{\sigma}}^F+\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\chi}}) \leqslant \mathbf{0}, \quad \forall t \geqslant \hat{t}. \tag{48}$$

In both Propositions 5 and 6, the load factor may or may not be regarded as a multiplier of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\chi}}$ and of the fictitious initial conditions (and, hence, of their linear effects $\hat{\boldsymbol{\sigma}}^F$), in view of the role of these quantities in the shakedown criteria 2 and 3. The latter alternative was preferred above in view of the computational methods discussed later. This very same remark, supplemented by the weak constitutive hypothesis that each plastic potential monotonically increases with an argument multiplier, justifies the fact that the scalar ω in (42) does not show up in (47). The transition from the strict inequality $\omega > 1$ to the loose inequality in (47b), in view of applications, should rigorously be accompanied by redefining α_s as the value separating the set of load factors ($\alpha < \alpha_s$) for which shakedown is ensured from the set of those ($\alpha > \alpha_s$) for which it is ruled out and by replacing max by sup in (47a).

The following remarks are intended as subsequent steps towards procedures of practical interest in applications.

(a) *Suboptimizations*. Let the initial conditions $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$, and the instant $\hat{t}$ be chosen *a priori* instead of being dealt with as optimization variables. With these additional constraints, maximization problems (47) and (48) yield suboptimal values and reduce to much simpler problems, i.e. respectively, to:

$$\alpha_s \geqslant \alpha_s^- \geqslant \beta^-(\hat{\mathbf{u}}_0,\dot{\hat{\mathbf{u}}}_0,\hat{t}) \equiv \max_{\alpha,\hat{\rho},\hat{\chi}} \{\alpha\} \text{ subject to (47b)} \tag{49}$$

$$\alpha_s \leqslant \alpha_s^+ \geqslant \beta^+(\hat{\mathbf{u}}_0,\dot{\hat{\mathbf{u}}}_0,\hat{t}) \equiv \max_{\alpha,\hat{\rho},\hat{\chi}} \{\alpha\} \text{ subject to (48b).} \tag{50}$$

Clearly, the solution of (49) is potentially more useful than that of (50), which provides unconservative information on α_s.

Natural choices of the fictitious initial conditions are: (i) homogeneous conditions ($\hat{\mathbf{u}}_0 = \mathbf{0}, \dot{\hat{\mathbf{u}}}_0 = \mathbf{0}$) which imply $\hat{\boldsymbol{\sigma}}^F(t) \equiv \mathbf{0}$ and eliminate the elastodynamic problem B of Section

3.1 ; (ii) actual conditions ($\hat{\mathbf{u}}_0 = \mathbf{u}_0, \dot{\hat{\mathbf{u}}}_0 = \dot{\mathbf{u}}_0$), which identify the fictitious process $\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^F$ with the elastodynamic response to the complete set of external actions.

However, in the special case of periodic loads $\mathbf{P}(t)$, a different choice becomes definitely more attractive as shown below.

(b) *Periodic excitation.* The fact that the actual initial conditions have no influence on shakedown (understood as boundedness in time of the dissipated energy) is physically well expected and quite apparent from the dynamic shakedown theory both in the classical and in the present context (see criteria of Section 4.1). In many engineering situations, the external actions are periodic and the structural response which practically matters consists of steady state periodic motion unaffected by initial conditions (the initial transient affected by the initial conditions, which are often uncertain, is soon or later damped off). Peculiar nonlinearities, such as geometric or/and physical (constitutive) instabilities, may cause non periodic chaotic responses [e.g. see Maier and Perego (1992)]. For linear elastodynamic responses, steady state periodic motion rigorously occurs asymptotically in time and can be uniquely determined under the following hypotheses (T being a finite excitation period):

$$\mathbf{P}(t+T) = \mathbf{P}(t), \quad \dot{\mathbf{u}}^T \mathbf{V} \dot{\mathbf{u}} > 0, \quad \forall \dot{\mathbf{u}} \neq \mathbf{0}. \tag{51}$$

Then, one can easily compute the special fictitious initial conditions such that, if actually imposed at $t = 0$, they would make the asymptotic, steady state periodic motion to start from $t = 0$ (*transient-suppressing* conditions, say $\mathbf{u}_0^P, \dot{\mathbf{u}}_0^P$). From this notion a simple path of reasoning adopted in the classical context [e.g. see Corradi and Maier (1973) and Maier and Novati (1990a)] and not duplicated here for brevity, leads to the following practically important statement.

Proposition 7. *Periodic excitation.* Under periodic loading, eqns (51a), the transient suppressing conditions and the time origin are optimal in both maximizations (47) and (48).

In other terms, setting $\hat{t} = 0$ and $\hat{\mathbf{u}}_0 = \mathbf{u}_0^P, \dot{\hat{\mathbf{u}}}_0 = \dot{\mathbf{u}}_0^P$, the optimizations (47) and (48) are to be performed in α, $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\chi}}$ alone; i.e. with reference to problems (49) and (50):

$$\beta^{+}(\mathbf{u}_0^P, \dot{\mathbf{u}}_0^P, 0) = \alpha_s^{+} ; \quad \beta^{-}(\mathbf{u}_0^P, \dot{\mathbf{u}}_0^P, 0) = \alpha_s^{-}. \tag{52}$$

(c) *Time removal.* As another step towards applications, without loss of generality, time can be eliminated as follows from the preceding optimization problems.

Starting form the elastodynamics stress history $\boldsymbol{\sigma}^E(t)$ for $t \geqslant \hat{t}$ preliminarly obtained by solving the linear problem (A), Section 3.1 [with data $\mathbf{P}(t)$, $\mathbf{u}_0^E = \mathbf{0}$, $\dot{\mathbf{u}}_0^E = \mathbf{0}$], a minimum *convex hull* H_j containing $\sigma_j^E(t)$ for $t > \hat{t}$ locally, in the stress space of each (j-th) element or Gauss point can be defined.

The free vibration response $\boldsymbol{\sigma}^F(t)$ either does not intervene (with homogeneous or transient-suppressing fictitious initial conditions) or can be ignored by choosing $\hat{t}$ sufficiently large in the suboptimizations. In the most general case, $\boldsymbol{\sigma}^F(t)$ can be dealt with as above specified for $\boldsymbol{\sigma}^E(t)$, like a term added to it when the fictitious initial conditions are not considered as optimization variables.

Let Γ_j be the boundary of H_j (or *elastic envelope*). Thus, because of the assumed convexity of both plastic potentials and yield functions (Section 2.2), all the preceding maximization problems can be transformed into relevant suboptimization problems by enforcing their constraints over $\boldsymbol{\sigma}^E \in \Gamma$ (or $\boldsymbol{\sigma}^E + \hat{\sigma}^F \in \Gamma$), Γ being the union of all $\Gamma_j (\Gamma = \cup_j \Gamma_j)$, instead of over all $t > \hat{t}$.

Let Γ_j^* denote the *set of vertices* of a convex *polyhedron* which contains the minimum convex hull H_j in the relevant (j-th) stress space. Then all the preceding maximization problems, including the suboptimization just mentioned are transformed into relevant suboptimizations by enforcing their constraints over $\boldsymbol{\sigma}^E \in \Gamma^*$ (or $\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^F \in \boldsymbol{\Gamma}^*$), Γ^* being the union of all Γ_j^*. The noteworthy computational gain now achieved is that the new problems are *mathematical programming problems* (with discrete numbers of both variables and constraints).

5. BOUNDS ON HISTORY-DEPENDENT QUANTITIES

5.1. *Bounding inequalities*

At present, the shakedown theory usually concerns both the original problem of determining the safety factor with respect to inadaptation and the related problem of achieving information, primarily upper bounds, on quantities which depend on the yielding history, in both cases by skipping the step by step solutions. Pioneering work on bounds in inelastic structural analysis was done by Martin (1964) and Ponter (1972, 1975b). The original distinction between direct and indirect bounds [e.g. see König and Maier (1981)] is superseded by unifying approaches [cf. Polizzotto (1982) and Maier and Novati (1990a)] of the kind adopted here. Accordingly, bounding properties are stated below first, and subsequently, in a single proof, they will be shown to flow from the fundamental inequality of Section 3.2. For convenience, it is recalled here from eqns (29b,c) and, for the special case of linear hardening, from (46), the meaning of the elastic strain energy $L_e(0)$ and of the stored energy $L_s(0)$, associated with the difference between actual and fictitious processes at the time origin $t = 0$ (subscript 0 for time-dependent quantities):

$$L_e(0) = \tfrac{1}{2}(\boldsymbol{\sigma}_0-\hat{\boldsymbol{\sigma}}_0)^T\mathbf{E}^{-1}(\boldsymbol{\sigma}_0-\hat{\boldsymbol{\sigma}}_0)+\tfrac{1}{2}(\dot{\mathbf{u}}_0-\dot{\hat{\mathbf{u}}}_0)^T\mathbf{M}(\dot{\mathbf{u}}_0-\dot{\hat{\mathbf{u}}}_0)$$

$$= \tfrac{1}{2}(\dot{\mathbf{u}}_0-\dot{\hat{\mathbf{u}}}_0)^T\mathbf{M}(\dot{\mathbf{u}}_0-\dot{\hat{\mathbf{u}}}_0)+\tfrac{1}{2}(\boldsymbol{\sigma}_0-\hat{\boldsymbol{\sigma}}_0^E)^T\mathbf{E}^{-1}(\boldsymbol{\sigma}_0-\hat{\boldsymbol{\sigma}}_0^E)-\tfrac{1}{2}\hat{\mathbf{p}}^T\mathbf{Z}\hat{\mathbf{p}}-(\boldsymbol{\sigma}_0-\hat{\boldsymbol{\sigma}}_0^E)^T\mathbf{E}^{-1}\mathbf{Z}\hat{\mathbf{p}} \quad (53a)$$

$$L_s(0) = \Psi_s(\boldsymbol{\eta}_0)-\Psi_s(\hat{\boldsymbol{\eta}})-(\boldsymbol{\eta}_0-\hat{\boldsymbol{\eta}})^T\hat{\boldsymbol{\chi}} = \tfrac{1}{2}(\boldsymbol{\eta}_0-\hat{\boldsymbol{\eta}})^T\mathbf{A}(\boldsymbol{\eta}_0-\hat{\boldsymbol{\eta}}). \quad (53b)$$

The second expression above given to the function $L_e(0)$ can be obtained using the definition of the fictitious stress response $\hat{\boldsymbol{\sigma}}(t)$ [eqn (27)], the fact that $\boldsymbol{\sigma}^E(0) = 0$ [eqns (23b), and eqns (26a,b)] which relate the self stresses $\hat{\boldsymbol{\rho}}$ to plastic strains $\hat{\mathbf{p}}$.

Often it is reasonable to assume that no plastic deformations exist at $t = 0$. Then in view of eqns (1) and (11): $\boldsymbol{\eta}_0 = \mathbf{0}$ (and, hence, $\Psi_s(0) = 0$); $\boldsymbol{\sigma}_0 = \mathbf{E}\mathbf{e}_0 = \mathbf{E}\mathbf{C}\mathbf{u}_0$. In this case, when the fictitious initial conditions are assumed to be coincident with the actual initial conditions, the r.h.s. of (53a) reduces to $\frac{1}{2}\hat{\boldsymbol{\rho}}^T\mathbf{E}^{-1}\hat{\boldsymbol{\rho}} = -\frac{1}{2}\hat{\mathbf{p}}^T\mathbf{Z}\hat{\mathbf{p}}$.

Proposition 8. *Bound on linear functions of* $\boldsymbol{\lambda}$. A linear combination with coefficients $\mathbf{w} \geqslant \mathbf{0}$ of the cumulative plastic multipliers $\boldsymbol{\lambda}(t)$ which measures the yielding of the available modes up to the time instant $\bar{t} > 0$, admits the following upper bound:

$$\mathbf{w}^T\boldsymbol{\lambda}(\bar{t}) \leqslant L_e(0)+L_s(0)+\mathbf{w}^T\boldsymbol{\lambda}(0) \quad (54a)$$

$$\text{if: } \boldsymbol{\Phi}(\hat{\boldsymbol{\sigma}}(t), \hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}-\mathbf{w}, \text{ over } 0 \leqslant t \leqslant \bar{t}. \quad (54b)$$

Proposition 9. *Bound on linear functions of* $\mathbf{p}$. A linear combination, with coefficients $\mathbf{r}$, of the plastic deformations $\mathbf{p}$ developed up to an instant $\bar{t}$, is bounded above by the inequality:

$$\mathbf{r}^T\mathbf{p}(\bar{t}) \leqslant L_e(0)+L_s(0)+\mathbf{r}^T\mathbf{p}(0) \quad (55a)$$

$$\text{if: } \boldsymbol{\Phi}(\hat{\boldsymbol{\sigma}}(t)+\mathbf{r}, \hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \text{ over } 0 \leqslant t \leqslant \bar{t}. \quad (55b)$$

Proposition 10. *Bound on linear functions of* $\boldsymbol{\eta}$. A linear combination, with coefficients $\mathbf{v}$, of the kinematic internal variables $\boldsymbol{\eta}$ at instant t is bounded above by:

$$-\mathbf{v}^T\boldsymbol{\eta}(\bar{t}) \leqslant L_e(0)+L_s(0)-\mathbf{v}^T\boldsymbol{\eta}(0) \quad (56a)$$

$$\text{if: } \boldsymbol{\Phi}(\hat{\boldsymbol{\sigma}}(t), \hat{\boldsymbol{\chi}}+\mathbf{v}) \leqslant \mathbf{B}, \text{ over } 0 \leqslant t \leqslant \bar{t}. \quad (56b)$$

Proposition 11. *Bound on plastic work*. The plastic work performed in the whole structural model up to the time t admits the bound:

$$\int_0^{\bar{t}} \boldsymbol{\sigma}^{\mathrm{T}} \dot{\mathbf{p}} \, \mathrm{d}\tau \leqslant \frac{1}{(q-1)} \cdot [qL_e(0) + L_s(0)] \tag{57a}$$

$$\text{if: } q > 1, \quad \boldsymbol{\Phi}(q\hat{\boldsymbol{\sigma}}(t), \hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \text{over } 0 \leqslant t \leqslant \bar{t}. \tag{57b}$$

Proposition 12. *Bound on energy* Ψ_s. The energy Ψ_s ($\boldsymbol{\eta}$) stored in the whole structure up to time $\bar{t}$ (because of rearrangements at the microscale represented by the kinematic internal variables $\boldsymbol{\eta}$) admits the bound:

$$\Psi_s(\bar{t}) = \int_0^{\bar{t}} \boldsymbol{\chi}^{\mathrm{T}} \dot{\boldsymbol{\eta}} \, \mathrm{d}\tau \leqslant \frac{1}{(1-s)} [L_e(0) + sL_s(0)] \tag{58a}$$

$$\text{if: } 0 < s < 1, \quad \boldsymbol{\Phi}(\hat{\boldsymbol{\sigma}}(t), s\hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \text{over } 0 \leqslant t \leqslant \bar{t}. \tag{58b}$$

Proposition 13. *Bounds on residual displacements*. Residual displacements $\mathbf{u}^p$ at time $\bar{t}$ are the displacements which would define the deformed configuration if the actual plastic strains $\mathbf{p}$ at time $\bar{t}$ were imposed as initial strains, statically on the structure. A linear combination with coefficients $\bar{\mathbf{P}}$ (*dummy loads*) of residual displacements at time $\bar{t}$, denoting by $\bar{\sigma}$ the elastostatic stress response to *dummy* loads $\bar{\mathbf{P}}$, is bounded above by:

$$\bar{\mathbf{P}}^{\mathrm{T}} \mathbf{u}^p(\bar{t}) \leqslant L_e(0) + L_s(0) + \bar{\boldsymbol{\sigma}}^{\mathrm{T}} \mathbf{p}(0) \tag{59a}$$

$$\text{if: } \boldsymbol{\Phi}(\hat{\boldsymbol{\sigma}}(t) + \bar{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \text{over } 0 \leqslant t \leqslant \bar{t}. \tag{59b}$$

Proofs. All the above statements are corollaries of Prop. 1, Section 3.2. In fact, they can be obtained from it by setting $\hat{t} = 0$ and choosing the perturbation variables so that, in the fundamental inequality (29a), the quantity to bound be isolated and related only to a function of available (and, hence, known) trial quantities. The trial quantities $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\chi}}$, $\hat{\mathbf{u}}_0$ and $\dot{\hat{\mathbf{u}}}_0$ are constrained by the constitutive inequality (28) specialized in turn to the conditions (54b)–(59b) under which the upper bounds (54a)–(59a), respectively, are valid.

Specifically, the suitable choices of the perturbation variables are easily seen to be as follows:

	q	s	$\mathbf{r}$	$\mathbf{v}$	$\mathbf{w}$
Prop 8	1	1	$\mathbf{0}$	$\mathbf{0}$	$\geqslant\mathbf{0}$
Prop 9	1	1	$\neq\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
Prop 10	1	1	$\mathbf{0}$	$\neq\mathbf{0}$	$\mathbf{0}$
Prop 11	>1	1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
Prop 12	1	$>0, <1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
Prop 13	1	1	$\bar{\boldsymbol{\sigma}}$	$\mathbf{0}$	$\mathbf{0}$

To prove Proposition 13 concerning residual displacements, it should be noticed that if $\boldsymbol{\rho}(\bar{t})$ are the self stresses owing to $\mathbf{p}(\bar{t})$, the virtual work principle requires that:

$$\bar{\mathbf{P}}^{\mathrm{T}} u^p(\bar{t}) = \bar{\boldsymbol{\sigma}}^{\mathrm{T}} [\mathbf{p}(\bar{t}) + \mathbf{E}^{-1} \boldsymbol{\rho}(\bar{t})]; \quad (\bar{\boldsymbol{\sigma}}^{\mathrm{T}} \mathbf{E}^{-1}) \boldsymbol{\rho}(\bar{t}) = \mathbf{0}. \tag{60}$$

Therefore, the linear combination of residual displacements (or work of dummy loads for them) to be bounded equals the quantity $\bar{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{p}$, on which the fundamental inequality (29), with the above specified choice of perturbation variables, directly provides an upper bound.

5.2. *Remarks on applications and bound optimizations*

First of all it is worth noting again that plastic deformations can be reasonably ruled out in the actual process at $t = 0$. This implies that $\boldsymbol{\lambda}(0) = \mathbf{0}$, $\mathbf{p}(0) = \mathbf{0}$ and $\boldsymbol{\eta}(0) = \mathbf{0}$ in eqns (54), (55a), (56a) and (59a).

In applying Props 8–10 and 13, a bound on a single component (say the i-th) of the relevant vector is likely to be of interest. Obviously, this is achieved by specializing the linear combination to bound, namely by setting to zero all coefficients but one and to, say, γ_i the coefficient of the component in point. It is worth noting, however, that γ_i represents an additional trial variable; in fact, setting $\gamma_i = 1$ is a legitimate choice but not necessarily the best one. Clearly the same can be said of a factor γ applied to any chosen vector ($\mathbf{r}$, $\mathbf{v}$ or $\mathbf{w}$) of linear combination coefficients.

Similarly, in Props 11 and 12 scalars q and/or s are additional trial parameters. The available trial parameters already present in shakedown analysis $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\chi}}$, $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$, and the new one γ (or γ_i) or q and/or s, depending on the quantity of interest, may be used to improve the bound by decreasing it. In fact, a bound may turn out to be much higher than the relevant actual quantity: then the information provided, though conservative, is hardly useful in practice. This motivates the optimizations (or suboptimizations) discussed below.

Consider, e.g. the upper bound (55), Prop. 9, on the j-th actual (generalized) plastic strain component at time $\bar{t}$, $p_j(t)$, singled out from vector $\mathbf{p}(\bar{t})$ by setting $\mathbf{r}^{\mathrm{T}} \equiv \{0 \ldots 0, r_j = \gamma, 0 \ldots 0\}$ and assuming $\mathbf{p}(0) = \mathbf{0}$.

Then, according to Prop. 9, the optimal upper bound is provided by the solution of the problem:

$$p_i \leqslant p_i^{\mathrm{opt}} = \min_{\substack{\gamma, \hat{\rho}, \hat{\chi} \\ \hat{\mathbf{u}}_0, \dot{\hat{\mathbf{u}}}_0}} \left\{ \frac{1}{\gamma} [L_{\mathrm{e}}(0) + L_{\mathrm{s}}(0)] \right\} \tag{61a}$$

$$\text{subject to:} \quad \gamma \geqslant 0, \quad \boldsymbol{\Phi}(\boldsymbol{\sigma}^{\mathrm{E}}(t) + \hat{\boldsymbol{\sigma}}^{\mathrm{E}}(t) + \hat{\boldsymbol{\rho}} + \mathbf{r}, \hat{\boldsymbol{\chi}}) \leqslant \mathbf{B}, \quad \text{over } 0 \leqslant t \leqslant \bar{t}, \tag{61b}$$

where $L_{\mathrm{e}}(0)$ and $L_{\mathrm{s}}(0)$ are specified by the expressions (53a) and (53b).

Now let us compare the above minimization problem (61) to the maximization problem (47) intended to optimize the lower bound on the safety factor with respect to inadaptation according to Prop. 5: it is noticed that the constraints are basically the same; the objective functions are both convex, quadratic in (61) and linear in (47) (in the variables $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\chi}}$).

The path of reasoning which led (47) to simpler suboptimizations and, finally, to a mathematical programming format, was unaffected by the nature of the objective function, and, hence, turns out to be applicable unaltered to simplify problem (61b) as well. For brevity this path will not be followed again in details, but only the main stages are recalled here for convenience: (a) suboptimizations by an *a priori* choice of the fictitious initial conditions; (b) in the case of periodic excitation, the *a priori* choice of the transient suppressing $\mathbf{u}_0^{\mathrm{p}}$, $\dot{\mathbf{u}}_0^{\mathrm{p}}$ is optimal, in the sense that do not deteriorate the optimal value; (c) time removal by recourse to minimum convex hulls Γ_i in the local stress spaces and reference to polyhedra enclosing those Γ_i in the same spaces in order to transform the problem in a fully algebraic one.

6. SHAKEDOWN ANALYSIS BY A KINEMATIC APPROACH

6.1. *Admissible yielding cycles*

The approach to be developed here is *kinematic* in the sense that it is based on suitably defined, fictitious, plastic deformation processes. In classical perfect plasticity this notion led to the theorem of Symonds and Neal (1951) and Koiter (1956), later shown to be *dual* to Melan's theorem through the duality formalism of mathematical programming (Maier, 1969) for piece-wise-linear yield condition. Its extension to dynamics was established by Corradi and Maier (1973, 1974). Extensions to associative nonlinear hardening plasticity have been presented in Comi and Corigliano (1991) and Polizzotto *et al.* (1991, 1993).

In view of the further generalization to the present nonassociative plasticity context, the notion of admissible cycle is preliminarily revisited below and defined in two different ways.

(a) *Admissible cycle of kind* φ (in symbols $\dot{\mathbf{p}}_\varphi$, $\dot{\boldsymbol{\eta}}_\varphi$) over the time interval $[t_1, t_2]$ will label henceforth a fictitious yielding process in which: (i) the flow rule is associated to the yield functions $\boldsymbol{\varphi}$, i.e.

$$\dot{\mathbf{p}}_\varphi = \frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\eta}}_\varphi = -\frac{\partial \boldsymbol{\varphi}^{\mathrm{T}}}{\partial \boldsymbol{\chi}} \dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi} \leqslant \mathbf{0}, \quad \boldsymbol{\varphi}^{\mathrm{T}} \dot{\boldsymbol{\lambda}} = \mathbf{0} \tag{62}$$

and (ii) the following relations are satisfied (the former means compatibility of the cumulative plastic strains developed along the cycle):

$$\Delta \tilde{\mathbf{p}}_\varphi \equiv \int_{t_1}^{t_2} \dot{\mathbf{p}}_\varphi(t)\,\mathrm{d}t = \mathbf{C}\Delta\tilde{\mathbf{u}}_\varphi, \quad \Delta\tilde{\boldsymbol{\eta}}_\varphi \equiv \int_{t_1}^{t_2} \dot{\boldsymbol{\eta}}_\varphi(t)\,\mathrm{d}t = \mathbf{0}. \tag{63}$$

Note that, as a consequence of eqns (62) and of the convexity of $\boldsymbol{\varphi}$ (Section 2), Hill's maximum principle holds for the relevant (fictitious) dissipation:

$$\dot{D}(\dot{\mathbf{p}}_\varphi, \dot{\boldsymbol{\eta}}_\varphi) \equiv \tilde{\boldsymbol{\sigma}}_\varphi^{\mathrm{T}} \dot{\mathbf{p}}_\varphi - \tilde{\boldsymbol{\chi}}_\varphi^{\mathrm{T}} \dot{\boldsymbol{\eta}}_\varphi \geqslant \boldsymbol{\sigma}^{*\mathrm{T}} \dot{\mathbf{p}}_\varphi - \boldsymbol{\chi}^{*\mathrm{T}} \dot{\boldsymbol{\eta}}_\varphi, \quad \forall \boldsymbol{\sigma}^*, \boldsymbol{\chi}^* \quad \text{such that} \quad \boldsymbol{\varphi}(\boldsymbol{\sigma}^*, \boldsymbol{\chi}^*) \leqslant \mathbf{0}. \tag{64}$$

(b) *Admissible cycle of kind* Φ (in symbols $\dot{\mathbf{p}}_\Phi$, $\dot{\boldsymbol{\eta}}_\Phi$) over the time interval $[t_1, t_2]$ will denote a fictitious yielding process such that: (i) the flow rule is associated to the plastic potential, i.e.

$$\dot{\mathbf{p}}_\Phi = \frac{\partial \boldsymbol{\Phi}^{\mathrm{T}}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\eta}}_\Phi = -\frac{\partial \boldsymbol{\Phi}^{\mathrm{T}}}{\partial \boldsymbol{\chi}} \dot{\boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\lambda}} \geqslant 0, \quad \boldsymbol{\Phi} \leqslant \mathbf{B}, \quad (\boldsymbol{\Phi} - \mathbf{B})^{\mathrm{T}} \dot{\boldsymbol{\lambda}} = \mathbf{0} \tag{65}$$

and (ii) the following equations hold:

$$\Delta \tilde{\mathbf{p}}_\Phi \equiv \int_{t_1}^{t_2} \dot{\mathbf{p}}_\Phi(t)\,\mathrm{d}t = \mathbf{C}\Delta\tilde{\mathbf{u}}_\Phi, \quad \Delta\tilde{\boldsymbol{\eta}}_\Phi \equiv \int_{t_1}^{t_2} \dot{\boldsymbol{\eta}}_\Phi(t)\,\mathrm{d}t = \mathbf{0}. \tag{66}$$

Similarly to (64) by virtue of (65) and of the convexity of $\boldsymbol{\Phi}$ (Section 2), the associated dissipation is characterized by Hill's maximum principle:

$$\dot{D}_\Phi(\dot{\mathbf{p}}_\Phi, \dot{\boldsymbol{\eta}}_\Phi) \equiv \tilde{\boldsymbol{\sigma}}_\Phi^{\mathrm{T}} \dot{\mathbf{p}}_\Phi - \tilde{\boldsymbol{\chi}}_\Phi^{\mathrm{T}} \dot{\boldsymbol{\eta}}_\Phi \geqslant \boldsymbol{\sigma}^{*\mathrm{T}} \dot{\mathbf{p}}_\Phi - \boldsymbol{\chi}^{*\mathrm{T}} \dot{\boldsymbol{\eta}}_\Phi, \quad \forall \boldsymbol{\sigma}^*, \boldsymbol{\chi}^* \quad \text{such that} \quad \boldsymbol{\Phi}(\boldsymbol{\sigma}^*, \boldsymbol{\chi}^*) \leqslant \mathbf{B}. \tag{67}$$

6.2. *Inadaptation theorems*

The word *inadaptation* means here the event contrary to shakedown, the lack thereof.

Proposition 14. *Sufficient condition for inadaptation.* The structure will not shakedown under the given load history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0, \dot{\mathbf{u}}_0$, if there is an admissible yielding cycle of kind φ, say $(\dot{\mathbf{p}}_\varphi, \dot{\boldsymbol{\eta}}_\varphi)$, starting at $t_1 > \hat{t}$, such that:

$$\int_{t_1}^{t_2} [\boldsymbol{\sigma}^{\mathrm{E}}(t) + \hat{\boldsymbol{\sigma}}^{\mathrm{F}}(t)]^{T} \dot{\mathbf{p}}_\varphi(t)\,\mathrm{d}t > \int_{t_1}^{t_2} \dot{\mathbf{D}}(\dot{\mathbf{p}}_\varphi, \dot{\boldsymbol{\eta}}_\varphi)\,\mathrm{d}t \tag{68}$$

for all $\hat{\boldsymbol{\sigma}}^{\mathrm{F}}(t)$, i.e. for any initial conditions $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$, and for all time instants $\hat{t}$.

Proof. Suppose, by contradiction with the thesis, that shakedown occurs under the given dynamic excitation $\mathbf{P}(t)$. Then, by virtue of Prop. 3, there exists some $\hat{t}$ and some fictitious process with time independent $\hat{\boldsymbol{\chi}}$, $\hat{\boldsymbol{\rho}}$ and initial conditions $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$ (and, hence, $\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^F + \hat{\boldsymbol{\rho}}$) such that (43) is satisfied. By identifying these $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\chi}}$ with $\boldsymbol{\sigma}^*$ and $\boldsymbol{\chi}^*$ in the maximum property (64) of an admissible cycle of kind φ and by integrating (64) over the time for $t_1 > \hat{t}$, we obtain the inequality:

$$\int_{t_1}^{t_2} (\hat{\boldsymbol{\sigma}}^T \dot{\tilde{\mathbf{p}}}_\varphi - \hat{\boldsymbol{\chi}}^T \dot{\tilde{\boldsymbol{\eta}}}_\varphi)\, dt \leqslant \int_{t_1}^{t_2} \dot{D}(\dot{\tilde{\mathbf{p}}}_\varphi, \dot{\tilde{\boldsymbol{\eta}}}_\varphi)\, dt. \tag{69}$$

Owing to the peculiar features (63) of the admissible yielding cycle and to the virtual work principle:

$$\int_{t_1}^{t_2} \hat{\boldsymbol{\rho}}^T \dot{\tilde{\mathbf{p}}}_\varphi\, dt = \hat{\boldsymbol{\rho}}^T \Delta \tilde{\mathbf{p}}_\varphi = \hat{\boldsymbol{\rho}}^T \mathbf{C} \Delta \tilde{\mathbf{u}}_\varphi = 0\,; \quad \int_{t_1}^{t_2} \hat{\boldsymbol{\chi}}^T \dot{\tilde{\boldsymbol{\eta}}}_\varphi\, dt = 0. \tag{70}$$

Through eqns (70), inequality (69) becomes;

$$\int_{t_1}^{t_2} (\boldsymbol{\sigma}^E(t) + \hat{\boldsymbol{\sigma}}^F(t))^T \dot{\tilde{\mathbf{p}}}_\varphi(t)\, dt \leqslant \int_{t_1}^{t_2} \dot{D}(\dot{\tilde{\mathbf{p}}}_\varphi, \dot{\tilde{\boldsymbol{\eta}}}_\varphi)\, dt \tag{71}$$

which holds for any admissible cycle of kind φ with $t_1 \geqslant \hat{t}$ and for the particular values $\hat{\sigma}^F$, $\hat{t}$ which derive from the shakedown necessary condition.

Inequality (71) is in contradiction with the hypothesis expressed by inequality (68), which holds for at least one admissible cycle of kind φ and for any $\hat{\boldsymbol{\sigma}}^F$, $\hat{t}$. (q.e.d.)

Proposition 15. *Necessary condition for inadaptation.* If the structure does not shakedown under the given loading history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, then there is some admissible yielding cycle of kind Φ, say $(\tilde{\mathbf{p}}_\Phi, \tilde{\boldsymbol{\eta}}_\Phi)$, starting at $t_1 > \hat{t}$, such that:

$$\xi \int_{t_1}^{t_2} [\boldsymbol{\sigma}^E(t) + \hat{\boldsymbol{\sigma}}^F(t)]^T \dot{\tilde{\mathbf{p}}}_\Phi(t)\, dt > \int_{t_1}^{t_2} \dot{D}(\dot{\tilde{\mathbf{p}}}_\Phi, \dot{\tilde{\boldsymbol{\eta}}}_\Phi)\, dt \tag{72}$$

for all $\hat{\sigma}^F(t)$, i.e. for any initial conditions $\hat{\mathbf{u}}_0$, $\dot{\hat{\mathbf{u}}}_0$, for any time instant $\hat{t}$ and scalar variable $\xi > 1$.

Proposition 15 can be restated, equivalently through formal logic, as a sufficient condition for shakedown in the way which follows. Note in passing that, similarly, Prop. 14 might be reformulated as a necessary condition for shakedown.

Proposition 16. *Sufficient condition for shakedown.* Shakedown will occur in the structure under the given loading history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, if there exist a fictitious free vibration response $\bar{\hat{\boldsymbol{\sigma}}}^F$, a time instant $\bar{\hat{t}}$, a scalar variable $\bar{\xi} > 1$, such that:

$$\bar{\xi} \int_{t_1}^{t_2} [\boldsymbol{\sigma}^E(t) + \bar{\hat{\boldsymbol{\sigma}}}^F(t)]^T \dot{\tilde{\mathbf{p}}}_\Phi(t)\, dt \leqslant \int_{t_1}^{t_2} \dot{D}(\dot{\tilde{\mathbf{p}}}_\Phi, \dot{\tilde{\boldsymbol{\eta}}}_\Phi)\, dt \tag{73}$$

for all admissible yielding cycles of kind Φ starting at $t_1 > \bar{\hat{t}}$.

Proof. It is proved below that condition (73) is a sufficient condition for SD by showing that the safety factor α_s is not less than 1 under the given loading history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$. To this purpose, consider the maximization problem (49) which provides a lower bound β on α_s, where $\hat{\boldsymbol{\sigma}}^F(t)$ and $\hat{t}$ are the *a priori* specified quantities $\bar{\hat{\boldsymbol{\sigma}}}^F(t)$, $\bar{\hat{t}}$, respectively, which appear in condition (73). In order to prove the proposition, we reintroduce the scalar factor ω which appeared in the shakedown sufficient condition (42) (see also the remarks after Proposition 6) and assign to it the value $\bar{\xi} > 1$ of the hypothesis (73). With this modification, maximization problem (49) is rewritten as:

$$\alpha_s \geq \alpha_s^- \geq \beta^-(\bar{\boldsymbol{\sigma}}^F, \bar{t}, \bar{\xi}) \equiv \max_{\alpha, \hat{\rho}, \hat{\chi}} \{\alpha\} \quad \text{subject to:} \tag{74a}$$

$$\boldsymbol{\Phi}(\bar{\xi}(\alpha\boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\sigma}}^F(t) + \hat{\rho}), \bar{\xi}\hat{\boldsymbol{\chi}}) \leq \mathbf{B}, \quad \forall t \geq \bar{t}; \quad \mathbf{C}^T\hat{\rho} = \mathbf{0}. \tag{74b}$$

The Lagrangian functional of the above optimization problem reads:

$$L = -\alpha + \int_{\bar{t}}^{t} \boldsymbol{\mu}^T[\boldsymbol{\Phi}(\bar{\xi}(\alpha\boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\sigma}}^F(t) + \hat{\rho}), \bar{\xi}\hat{\boldsymbol{\chi}}) - \mathbf{B} + \mathbf{d}]\,d\tau + \int_{\bar{t}}^{t} \mathbf{v}^T\mathbf{C}^T\hat{\rho}\,d\tau, \tag{75}$$

where $\boldsymbol{\mu}$ and $\mathbf{v}$ are vectors of Lagrange multipliers and $\mathbf{d}$ denotes a vector of positive slack variables which transforms the inequality constraint into an equality. The Euler–Lagrange optimality conditions for problem (74) can be computed as follows from functional (75), denoting for brevity by $\hat{\boldsymbol{\sigma}}$ the sum $\alpha\boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\sigma}}^F(t) + \hat{\rho}$:

$$\int_{\bar{t}}^{t} \bar{\xi}\boldsymbol{\sigma}^{E^T} \frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{\sigma}}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}})\boldsymbol{\mu}\,d\tau = 1 \tag{76a}$$

$$\int_{\bar{t}}^{t} \left[\bar{\xi}\frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{\sigma}}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}})\boldsymbol{\mu} + \mathbf{C}\mathbf{v}\right] d\tau = \mathbf{0} \tag{76b}$$

$$\int_{\bar{t}}^{t} \bar{\xi}\frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{\chi}}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}})\boldsymbol{\mu}\,d\tau = 0 \tag{76c}$$

$$\mathbf{C}^T\hat{\rho} = \mathbf{0} \tag{76d}$$

$$\boldsymbol{\Phi}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}}) \leq \mathbf{B}, \quad \boldsymbol{\Phi}^T\boldsymbol{\mu} = 0, \quad \boldsymbol{\mu} \geq \mathbf{0}. \tag{76e,f,g}$$

Notice now that the following vectors:

$$\dot{\mathbf{p}}_\Phi \equiv \frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{\sigma}}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}})\boldsymbol{\mu}, \quad \dot{\boldsymbol{\eta}}_\Phi \equiv -\frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{\chi}}(\bar{\xi}\hat{\boldsymbol{\sigma}}, \bar{\xi}\hat{\boldsymbol{\chi}})\boldsymbol{\mu} \tag{77}$$

define an admissible yielding cycle of kind Φ in the time interval $[t_1 = \bar{t}, t_2 = t]$. In fact variables $\dot{\mathbf{p}}_\Phi$, $\dot{\boldsymbol{\eta}}_\Phi$ satisfy the flow rule associated with functions $\boldsymbol{\Phi}$ (77), (76e–g) and relations (76b,c), which coincide with (66a,b) provided that $\mathbf{v}$ is interpreted as a vector of displacement rates.

Consider now the above defined particular admissible yielding cycle of kind Φ defined by (77). In view of what precedes the inequality (73) of the hypothesis must hold also for the above cycle, namely:

$$\bar{\xi}\int_{\bar{t}}^{t} \bar{\boldsymbol{\sigma}}^{E^T}\dot{\mathbf{p}}_\Phi\,d\tau \leq \bar{\xi}\int_{\bar{t}}^{t} \hat{\boldsymbol{\sigma}}^T\dot{\boldsymbol{\eta}}_\Phi\,d\tau + \bar{\xi}\int_{\bar{t}}^{t} \hat{\boldsymbol{\chi}}^T\dot{\boldsymbol{\eta}}_\Phi\,d\tau \tag{78}$$

where $\bar{\boldsymbol{\sigma}}^E(t) = \boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\sigma}}^F(t)$ and $\hat{\boldsymbol{\sigma}}(t) = (\alpha\boldsymbol{\sigma}^E(t) + \bar{\boldsymbol{\sigma}}^F(t) + \hat{\rho})$. Making use of conditions (76b,c) and noting that $\hat{\rho}$, $\hat{\boldsymbol{\chi}}$ are time independent and that $\hat{\rho}$ is self-equilibrated, the inequality (78) can be transformed into the following form:

$$\int_{\bar{t}}^{t} \bar{\xi} \boldsymbol{\sigma}^{E}(t)^{T} \dot{\mathbf{p}}_{\Phi} \, d\tau \leqslant \alpha \int_{\bar{t}}^{t} \bar{\xi} \boldsymbol{\sigma}^{E}(t)^{T} \dot{\tilde{\mathbf{p}}}_{\Phi} \, d\tau. \tag{79}$$

As a consequence of eqns (79) and (76a) it follows that α is no less than one. Since the above Euler–Lagrange relations of the maximization problem (74) are always satisfied by the optimal solution, one can write the following chain of inequalities:

$$\alpha_s \geqslant \alpha_s^- \geqslant \beta^{-1}. \tag{80}$$

The inequalities (80) state that the shakedown limit α_s for the structure under the given loading history $\mathbf{P}(t)$ and initial conditions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$ is no less than one. Hence shakedown is ensured and Proposition 16 is proved. Owing to the equivalence between Propositions 15 and 16, the latter implies the former to be satisfied. (q.e.d.).

It is worth noting that the scalar $\bar{\xi}$ disappears from the argument of the gradient $\partial \boldsymbol{\Phi}^T / \partial \boldsymbol{\sigma}$ and $\partial \boldsymbol{\Phi}^T / \partial \boldsymbol{\chi}$ in eqns (76) and (77) if the plastic potentials $\boldsymbol{\Phi}$ are assumed positively homogeneous of order 1, according to a frequently adopted weak hypothesis.

7. CONCLUSIONS

In what precedes reference was made to material behaviour described by generally nonassociative elastic–plastic (time-independent) constitutive laws which exhibit nonlinear hardening governed by internal variables occurring in pairs (according to the so-called generalized standard and non-standard elastoplasticity).

A notion crucial to the purposes of the present paper is the reduced yield domain proposed in Maier's earlier shakedown theory in nonassociative perfect plasticity.

Semidiscretization (in space) has been adopted by multifield finite element modeling, so that vectors of generalized variables (in Prager's sense, occurring in pairs) govern the evolution of the discretized solid or structure considered.

On this basis and in the assumed absence of geometric effects, the following contributions have been presented to the shakedown theory of elastoplastic structural dynamics.

(a) A further extension, in terms of more general constitutive laws, of the static shakedown theorem of Melan, extended to dynamics by Ceradini. The present extension, studied as for the quasi-static context in a separate paper, materializes in distinct sufficient and necessary shakedown conditions, which yield lower and upper bounds on the shakedown load factor.
(b) A further parallel generalization of the kinematic Koiter's theorem and of its earlier extension to dynamics by Corradi and Maier. The present results consist of distinct sufficient and necessary conditions for inadaptation.
(c) Upper bounds on various post-shakedown quantities, as extended versions of bounding inequalities supposed to be most promising among those available in the literature for narrower constitutive contexts.
(d) Both the optimization of the upper bounds on history-dependent quantities and the computation of bounds on the shakedown limits are shown to be amenable to mathematical programming problems.

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